

# pvar: the calculation and applications of $p$ -variation

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## Abstract

In this vignette we will give a brief introduction to the  $p$ -variations. Later on, we will give some illustrative examples of the main functions in *pvar* package.

## 1 Introduction

Let start with the definitions. Originally  $p$ -variation is defined for a functions. For a function  $f : [0, 1] \rightarrow R$  and  $0 < p < \infty$   $p$ -variation is defined as

$$v_p(f) = \sup \left\{ \sum_{i=1}^m |f(t_i) - f(t_{i-1})|^p : 0 = t_0 < t_1 < \dots < t_m = 1, m \geq 1 \right\}.$$

Suppose  $X = \{X_i\}_{i=0}^n$  is any real-value sequence of numbers. We will call such sequence a *sample*, whereas  $n$  will be referred to as a *sample size*. The  $p$ -variation of the sample could be expressed as

$$v_p(X) = \max \left\{ \sum_{i=1}^k |X_{j_i} - X_{j_{i-1}}|^p : 0 = j_0 < \dots < j_k = n, k = 1, \dots, n \right\}.$$

The set of index  $j_0, \dots, j_k$  that achieves the maximum is refereed as supreme partition.

Lots and detailed information on the known properties of  $p$ -variation can be found in the books of Norvaiša and Dudley (see [2] and [3]). There are formulated and proved a lot of the properties of  $p$ -variation. In addition, the comprehensive bibliography of  $p$ -variation is include.

The core functions of the package is based on algorithm described in [1]. The calculation of  $p$ -variation was inspired by paper [4], where authors Norvaiša and Račkaskas presented necessary and sufficient conditions for the convergence in law of partial sum processes in  $p$ -variation norm. Based on the result of this paper, Račkaskas proposed the method of testing sample of multiple shift in mean. This test is also implemented in this package.

The algorithm is written in C++, therefore it is quite fast. It is worth noting, that *pvar* package could be used to calculate  $p$ -variation for any piecewise monotone functions and to investigate general properties of  $p$ -variation.

## 2 Main function

The main function in *pvar* package is *pvar* (that a surprise). It calculates the supreme partition and the value of  $p$ -variation. The main usage of *pvar* function goes as follows

```
N <- 1000
p <- 2
x <- rwiener(N)
pv <- pvar(x, p)
pv

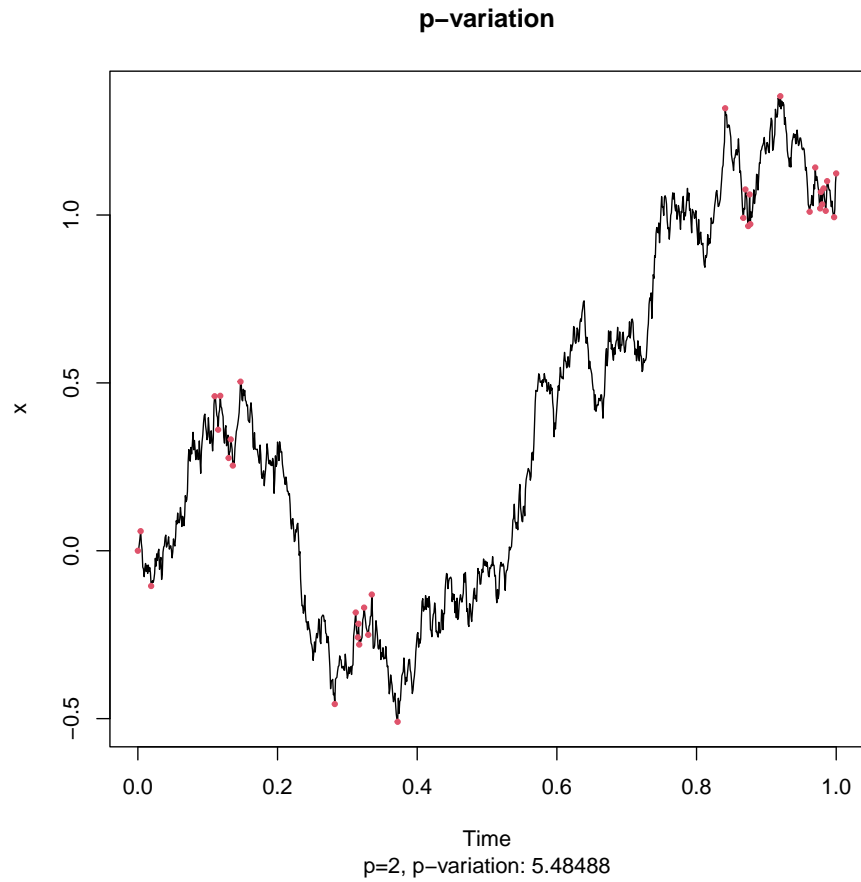
## p-variation
## 5.484877

summary(pv)

## The summary of p-variation:
## Value: 5.485, p = 2
## Data: x, n = 1000 (+1)
##
## Data vector (n=1001): 0, 0.001871, 0.01904, 0.03736, 0.05838, 0.0224, ...
## Partition has 36 points: 1, 5, 20, 111, 116, 119, ...
```

Moreover, the data and supreme partition could be illustrated with *plot* function

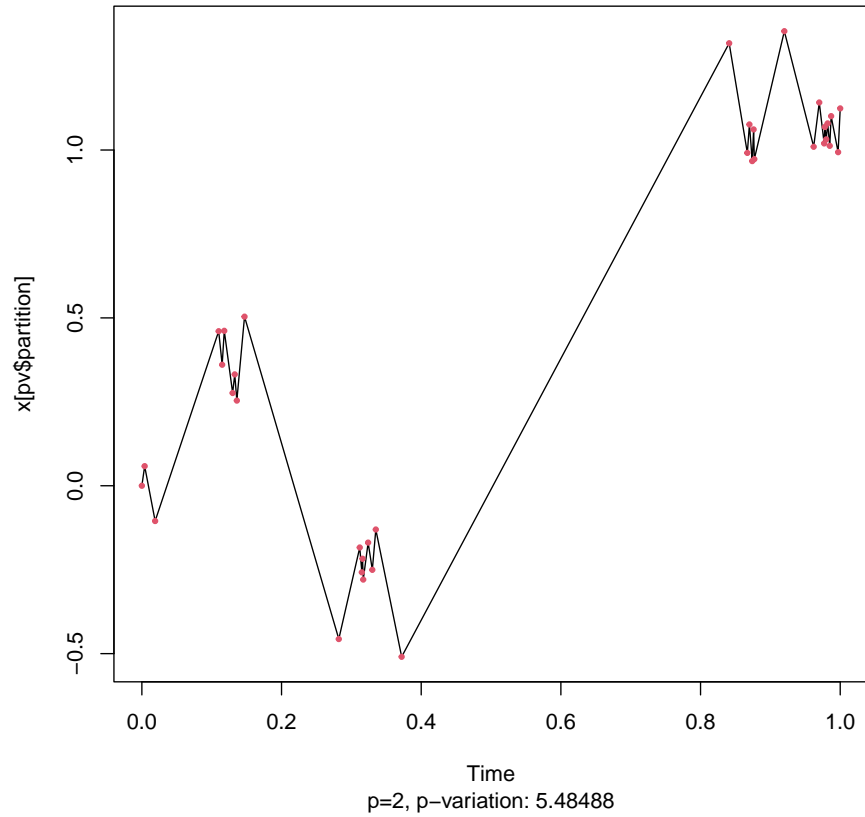
```
plot(pv)
```



The points in supreme partition (the red points) are actually used in  $p$ -variation. All the other points are redundant and could be dropped out with out any effect to  $p$ -variation. This fact is illustrated in the example below.

```
pv.PP = pvar(x[pv$partition], TimeLabel=time(x)[pv$partition], 2)
plot(pv.PP, main='The same pvar without redundant points')
```

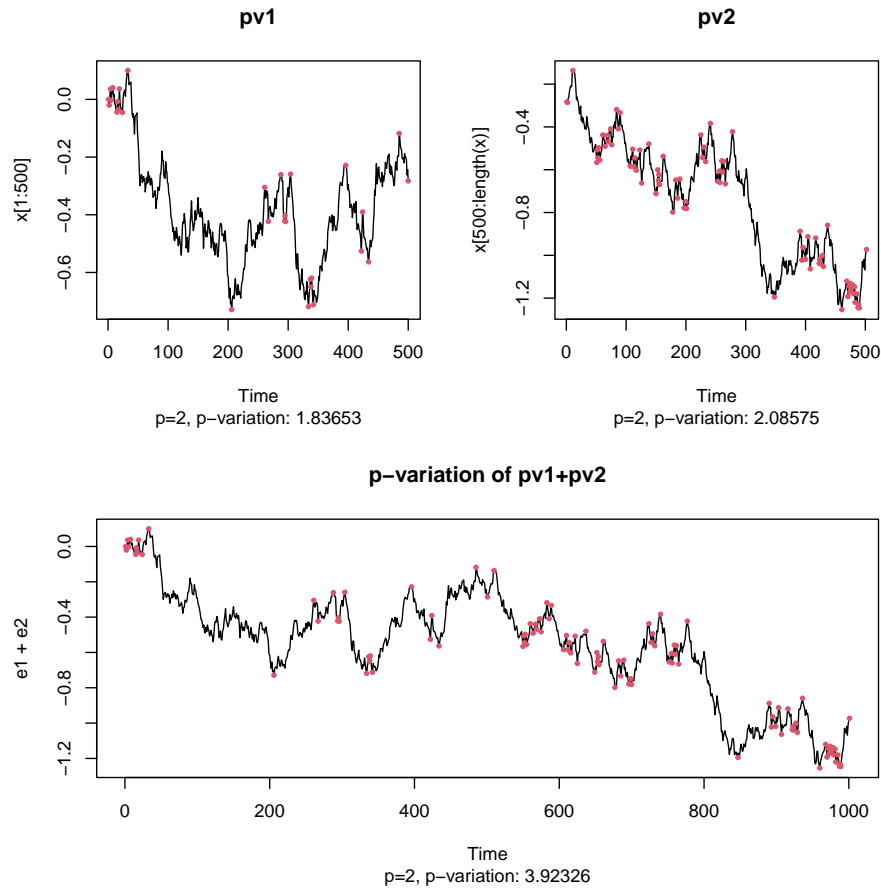
### The same pvar without redundant points



Moreover,  $p$ -variation on different intervals could be merged into unified sample. Let look at the example

```
x = rwiener(1000)
pv1 = pvar(x[1:500], 2)
pv2 = pvar(x[500:length(x)], 2)
sum_pv1_pv2 = pv1 + pv2
```

The illustration of this looks like this



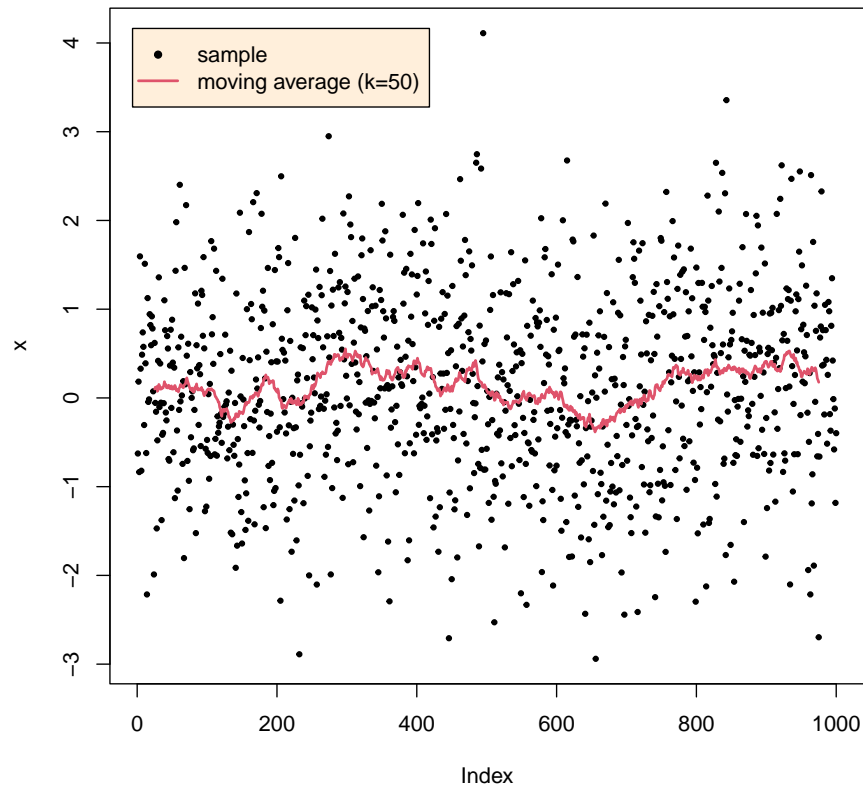
### 3 PvarBreakTest

A. Račkauskas has proposed a method of detection of multiple changes in a sequence of independent observations. This method is based on results presented in [4]. This method is realized in this package under the name of Pvar Break Test (or *PBT* for short). It is implemented in *PvarBreakTest* function.

Lets consider the example. Let construct the data  $x$  with multiple shifts in mean.

```
set.seed(1)
MiuDiff <- 0.3
x <- rnorm(250*4, rep(c(0, MiuDiff, 0, MiuDiff), each=250))
```

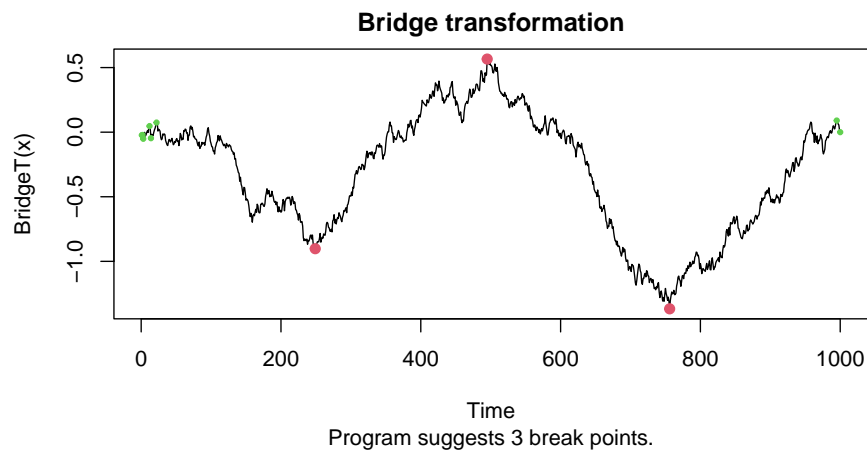
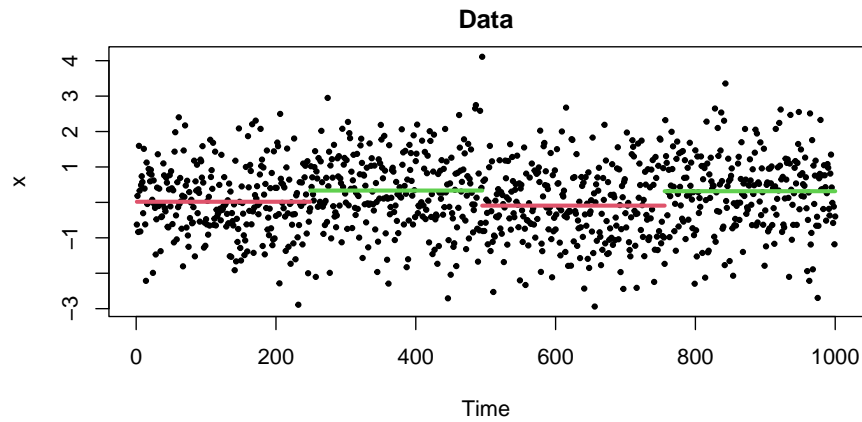
Original data, with several shifts in mean



From the original data, it is not obvious if there are any breaks in sample. To test whether there is a multiple break, we can apply *PvarBreakTest* test. The results are remarkable good. Look below.

```
xctest <- PvarBreakTest(x)
xctest

##          PvarBreakTest
##
## H0: no structural change
## Results: H0 is rejected at the confidence level of 0.05.
## Data: x, n=1000.
## The output of the test:
##      Statistics Critical Value      alpha      ~p.value
##      24.04223      12.76930      0.05000      0.00730
```



## References

- [1] V. Butkus, R. Norvaiša. Lith Math J (2018). <https://doi.org/10.1007/s10986-018-9414-3>
- [2] R. M. Dudley, R. Norvaiša. An Introduction to  $p$ -variation and Young Integrals, Cambridge, Mass., 1998.
- [3] R. M. Dudley, R. Norvaiša. Differentiability of Six Operators on Nonsmooth Functions and  $p$ -variation, Springer Berlin Heidelberg, Print ISBN 978-3-540-65975-4, Lecture Notes in Mathematics Vol. 1703, 1999.
- [4] R. Norvaiša, A. Račkauskas. Convergence in law of partial sum processes in  $p$ -variation norm // Lth. Math. J., Vol. 48, No. 2, 212–227, 2008.

- [5] J. Qian. The  $p$ -variation of partial sum processes and the empirical process  
// Ph.D. thesis, Tufts University, 1997.